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LETTER TO THE EDITOR

Infinite susceptibility at high temperatures in the Migdal–Kadanoff scheme

Miron Kaufman and Robert B Griffiths
Physics Department, Carnegie-Mellon University, Pittsburgh, PA 15213, USA

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Abstract. The Migdal–Kadanoff real-space renormalisation group scheme applied to the Ising and classical XY (and probably other) ferromagnetic models yields an infinite susceptibility at all temperatures above the critical temperature for a hypercubic lattice of dimension $d \geqslant 2$.

The renormalisation group scheme introduced by Migdal (1975) and further developed by Kadanoff (1976) has been applied to a large number of problems of phase transitions on lattices. (See, for example, Berker *et al* (1978), Kaufman *et al* (1981), Yeomans and Fisher (1981), and additional references given in these papers.)

We wish to point out that this scheme leads to singularities in the free energy as a function of magnetic field h at $h = 0$ at all temperatures *above* the critical temperature for certain ferromagnetic lattice models. Such behaviour is decidedly unphysical, and it can be shown rigorously (Israel 1976) that for at least some of these models (on a Bravais lattice), the free energy is analytic in h near $h = 0$ at sufficiently high temperatures.

From the viewpoint of the standard renormalisation group analysis, the singularity in question arises from the fact that the infinite-temperature (zero interaction) fixed point is unstable with respect to flows towards fixed points at $h = \pm\infty$, and indeed the form of the singularity is given correctly by the usual analysis (Niemeijer and Van Leeuwen 1976). However, the existence of an unstable fixed point is, in itself, no guarantee that a singularity actually occurs, since the amplitude of the singular term may in fact vanish. For examples, see Kaufman *et al* (1981) and Nelson and Fisher (1975). In the case of the Migdal–Kadanoff scheme, the fact that the recursion equations are exact for a model on a somewhat unphysical but nonetheless well defined 'hierarchical lattice' (Berker and Ostlund 1979, Bleher and Zalys 1979, Kaufman and Griffiths 1981) makes it possible to prove the existence of a singularity in the case of the Ising and classical XY models. The source of the singularity can, in addition, be traced to the peculiar geometrical properties of the corresponding hierarchical lattice.

As an example, consider the Migdal–Kadanoff scheme for an Ising model in $d = 2$ dimensions with a linear scale change of $b = 2$. The corresponding 'diamond hierarchical lattice' is generated iteratively by assembling four zero-order or primitive bonds, figure 1(a) to form a diamond (b) or bond of order 1. Four of these are then assembled in an identical manner, figure 1(c), to form a diamond of diamonds or bond of order 2, and so on *ad infinitum.* An Ising spin $\sigma_i = \pm 1$ is associated with the ith vertex,

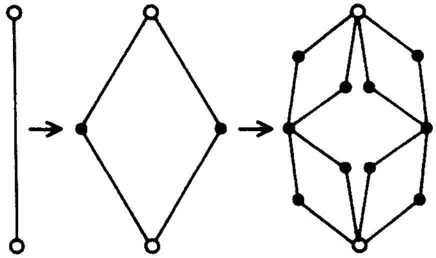

Figure 1. Construction of the diamond hierarchical lattice.

and the dimensionless Hamiltonian is given by

$$-\frac{\mathcal{H}}{kT}=\sum_{\langle ij\rangle}[J\sigma_i\sigma_j+h(\sigma_i+\sigma_j)],\tag{1}$$

with the sum over all pairs of sites at the opposite ends of primitive bonds.

The iteration procedure for calculating the partition function consists in successively summing the Boltzmann weight over the spins at sites of coordination number 2, 4, 8, ... At each stage one obtains an effective Hamiltonian of the type (1) for the remaining spins. The resulting transformation has three fixed points on the $h=0$, $J\geqslant 0$ axis: $J=\infty$ (zero temperature), $J=J_c$ (critical point) and $J=0$ (infinite temperature). The last is a sink for flows originating on the J axis for $J<J_c$, but is unstable with respect to flows along the h axis to the $J=0$, $h=\pm\infty$ fixed points.

The case $J=0$ (a decoupled system) is easily analysed. The dimensionless free energy f per bond is given by the uniformly convergent series

$$f(h)=2\sum_{n=1}^{\infty}4^{-n}\ln(2\cosh 2^n h)\tag{2}$$

and thus the susceptibility for $h\neq 0$ by

$$\chi(h)=f''(h)=2\sum_{n=1}^{\infty}(\cosh 2^n h)^{-2}.\tag{3}$$

An elementary analysis shows that χ exhibits a logarithmic divergence as h goes to zero:

$$\chi(h)=-2(\ln|h|)/\ln 2+\mathrm{O}(1).\tag{4}$$

We note in passing that this singularity arises from the classical Yang and Lee (1952) mechanism of zeros of the partition function on the imaginary h axis 'pinching' the origin in the thermodynamic limit. Indeed, the dense set of zeros on the imaginary axis produces a natural boundary for the analytic function $f(h)$ in the half plane $\mathrm{Re}(h)\geqslant 0$, somewhat reminiscent of the situation in random ferromagnets (Griffiths 1969).

For $J>0$ (i.e. finite temperature) we have been unable to carry out a corresponding analysis in the complex h plane. However, the divergence of χ at $h=0$ for all $J<J_c$ follows at once from the GKS inequalities (Griffiths 1972): for any $h>0$ the magnetisation can only increase as J increases. Note that this argument makes essential use of the fact that the Migdal–Kadanoff scheme is 'realisable' on a specific lattice to which standard inequalities can be applied; a similar statement would *not* be possible for an arbitrary approximate renormalisation group scheme. A more involved analysis (of which we shall not present the details) shows that the singularity continues to be logarithmic as $h \to 0$ for J between 0 and J_c.

The hierarchical lattice also provides insight as to the source of the unphysical divergent susceptibility. A Hamiltonian of the form (1) provides a magnetic field of magnitude qh at a site of coordination number q, and on the hierarchical lattice q is unbounded, though the number of sites with high coordinate number is relatively small. This observation suggests a remedy for the infinite χ: let the magnetic field be applied only to sites with the minimum coordination of $q = 2$. While this remedy works at infinite temperature ($J = 0$), it fails for $J > 0$, because the very first iteration of the usual renormalisation procedure produces a finite effective field at the ends of each bond of order 1. Consequently the logarithmic divergence of χ is again present for $0 < J < J_c$, even when a field is applied to sites of minimum coordination number.

The preceding analysis can be generalised to a number of other situations. Consider an Ising model with interactions (1) on a hierarchical lattice constructed by assembling B bonds, of which L are attached to the top and L to the bottom vertices of the bond of next higher order ($B = 4$, $L = 2$ in figure 1). For $J = 0$ the dominant susceptibility singularity is of the form

$$A|h|^p, \tag{5}$$

where A is, in general, a constant plus a periodic function of $\ln|h|$, and

$$p = (\ln B/\ln L) - 2. \tag{6}$$

If p is an even integer a term proportional to $h^p \ln|h|$ must be added to (5). For $p \leq 0$ the GKS inequalities can be used to show that the susceptibility continues to diverge for $0 < J < J_c$. Since

$$p = (2-d)/(d-1) \tag{7}$$

(independent of the linear scale change b) for the Migdal–Kadanoff scheme applied as an approximation to a d-dimensional hypercubic lattice, we see that a divergent susceptibility is present for $d \geq 2$.

A similar analysis can be carried out at infinite temperature ($J = 0$) for various other models, such as n-vector models. However, the GKS inequalities needed to demonstrate a divergent χ at finite temperatures have only been proved for $n = 2$ (Monroe and Pearce 1979).

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